AN EXPLORATION OF STUDENT RESPONSES TO THE MORE DEMANDING KÜCHEMANN TEST ITEMS

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Understanding symbolic notation is usually considered crucial to the study of mathematics. One significant aspect that has emerged from research into this understanding involves the meanings students give to pronumerals (see for example, Collis (1975), Pegg and Redden (1990)). Küchemann (1981), in particular, has identified six different interpretations for the meaning attributed to letters by junior secondary students. These interpretations have been grouped into four levels representing a hierarchy of understanding.

The purpose of this study was to explore students' responses to three of the more demanding Küchemann test items which were identified after an initial sample of 278 students were given the entire set of test items. Twenty one students were then interviewed and the reasons for their responses to these questions were analysed. Students' responses and associated reasons could be divided into two distinct categories, depending upon the mental processes involved in answering these questions. The distinguishing factor was found to be the ability to take into account the 'range of possibilities and limitations' associated with relationships involving letters.

Fundamental to the study of algebra (in terms of generalised arithmetic) is the meaningful interpretation of its symbolism. For many students, however, this conceptualisation merely consists of the manipulation of letters without any real understanding of what the letters actually represent.

Identifying levels of understanding in relation to the meanings given to symbols by students has been the subject of many studies (see for example Küchemann (1981), Booth (1984), Pegg and Redden (1990), Coady and Pegg (1991), Quinlan (1992)). In all of these studies it has been generally accepted that the highest level of understanding is that of interpreting a letter as a variable. Interpretation at this level requires higher-order mental functioning, where thought processes are not confined to seeing relationships but rather on focusing on the nature of the relationships. One aspect of this would occur when allowances can be made for any 'possibilities' and consequent 'limitations' inherent within the governing principles of the problem/question.

Initally, 278 first year tertiary mathematics students were scored on the Küchemann test items. An analysis of these results revealed that those questions involving the consideration of possibilities relating to the value of a pronumeral proved very demanding in terms of cognitive functioning. These questions were:

* Question 1. *Which is larger, $2n$ or $n + 2$? Explain.*

* Question 11. *What can you say about c if $c + d = 10$ and c is less than d?*

* Question 12. $L + M + N = L + P + N$ *is*
always sometimes (when) never true?
Circle your answer.

(* These question numbers refer to those on the original test paper.)

It was decided to consider, in depth, the range of responses to these three questions. In particular, the following issues were addressed:

1. Is there any consistency among groups of responses?
2. If these different groups of responses exist, what are their characteristic features?

This work extends the ideas offered by Küchemann as it is an attempt to explore the subtleties in the answers offered by students.

DESIGN

On analysing all 278 written test responses, the percentage of students incorrect on each of the abovementioned three questions were 79%, 64% and 50% respectively. In order to examine the reasons for these poor performances, twenty one students were then interviewed.

During the interview, each student was asked to attempt each of the questions previously mentioned while verbalising their thoughts. If further clarification of their reasoning was needed, 'how and why' questions were asked by the interviewer.

RESULTS AND DISCUSSION

Two distinct categories of responses could be identified for each of the questions. Broadly, these were:

Category a) responses gave no indication of an awareness of any underlying condition restricting the variable, i.e., reasoning was confined to manipulating the terms in the given system; and,

Category b) responses indicated that account was taken of various conditions, i.e., 'possibilities' were allowed for and the associated 'limitations' were determined.

An analysis of the responses to each question in each of these categories now follows.

Which is the larger, 2n or n + 2 ? Explain

Category a) responses to this question concentrated on the operations, multiplication and addition, resulting in the conclusion "2n". Sometimes this conclusion was verified by the substitution of one positive value of 'n'. Even if the interviewer probed, *"is this true for all values of n"*, the overriding consideration was still on the operations. The following extract characterises this type of response:

I: What is your answer to question 1?
J: *2n.*
I: Why?
J: *Because it is multiply ..... 2 x n, whereas the other is only plus.*
I: And is 2n larger for all values of n?
J: *Yes it would be. For example if n = 3, 2 x 3 = 6, but 3 + 2 = 5, and 6 is bigger than 5.*

Category b) responses to this question were characterised by the realisation that in order to determine the larger of the two expressions, the value of n needed to be considered. This was usually the first statement made by the student. This was then followed by the substitution of one, two of three values of n which were all in the vicinity of n = 2. Students making one substitution only, invariably chose n = 1, generalising this to "2n < n + 2 for n < 1" or chose n = 3 leading to the conclusion "2n > n + 2 for n > 3". Where two substitutions were made, values of n less than 1 and values greater than 3 led to the same conclusions as mentioned previously. These responses do not necessarily preclude the substitution n = 2 which may have occurred 'in the student's head'. The following is typical of the highest level of reasoning shown in this question:

I: How do you do question 1?
M: *Relating the two variables between the two expressions, the value of n determines whether or not 2n or n + 2 is larger. For example, if n = 1, 2n would be smaller than n + 2 whereas in the example of say n = 3, 2n would equal 6 and n + 2 would equal 5 so therefore the variable n determines whether 2n is larger or smaller than n + 2.*
I: Could we write down a statement that qualifies which would be the larger?
M: *If n is less than 2, 2n equals ..... , if n is less than 1, 2n is less than n + 2* [writes down 2n < n + 2 n < 1]; *if n = 2, 2n equals n + 2,* [writes down if n = 2, 2n = n + 2], *[if] n is greater than 2, 2n is greater than n + 2* [writes down if n > 2 , 2n > n + 2].

This student's response indicates that a complete overview of the problem was well within his/her grasp and accurate conclusions were drawn.

What can you say about c if c + d = 10, and c is less than d?

Category a) responses to this question were typified by the manipulation of symbols, similar to those techniques used when solving equations. Hence the most common response here was "c = 10 - d". Others in this category did try to incorporate the second part of the question ('c is less than d') which resulted in the expression "c < 10 - d". For example:

I: How could we work out an expression involving c?
E: *c plus d equals 10* [writes c + d = 10], *c is less than d* [writes c < d] .... could we write it like that? [writes c = 10 - d, c < 10 - d]
I: c is less than d and c is less than 10 - d, are they the same statement?
E: *No.*
I: What do we have to do now?
E: *I think maybe we should plot some numbers in.*
I: OK lets try that.
E: *Because it equals 10, we have to choose numbers less than 10......I really don't have any idea.*

Category b) responses again showed an understanding that some underlying condition on the value of c was needed. All students' responses in this category mentioned c as having a "boundary" value or that c was the "half-way" mark. Attempts to identify the answer can be considered within three groups of responses.
The first group in this category chose one value of c usually c = 4. After this value was seen to satisfy the conditions in the question, no further values were substituted and "c =4" represented the conclusion drawn.

Responses still characteristic of this category but representing a slightly higher level of understanding, again concentrated on values of c, but a systematic list such as "c =0, 1, 2, 3, 4" or "c goes from 0 - 4" was given. Under prompting, students were usually able to provide the verbal statement 'c must be less than 5'.

The third group were able to determine the critical value of c, after which a conclusion was drawn, as the next abstract shows:

I: What can you write down about c?
C: *First of all, because c is less than d, I actually write it down graphically* [writes c < d]. *Now because you have c + d = 10, that* [points to c < d] *means d must be greater than c. So for any number c, c is less than d so therefore c is less than 5* [writes c < 5].
I: Why 5?
C: *Because you have 10 as a number and because d is greater than c, then c cannot be greater than 5 because 5 is the half way mark, so you have 5 + 5, that will be 10, but it says c is less than d, then it cannot be 5 but it could be 4.9 plus 5.1 as d.*

L + M + N = L + P + N is

always   sometimes (when)   never true

Circle your answer.

While this question is similar in nature to the other two questions, in that there existed an underlying constraint on the variables, this constraint was not of the same style. Certainly here, a range of possibilities needs to be considered, but it is more general in nature and hence, not as difficult as the first two questions. That is, students need only to be able to accept that even though M and P can vary over a range of values there are situations in which equality is a possibility.

Category a) responses formed two groups. The first group answered this question with 'never', occasionally supported with a statement such as "$M \neq P$". Prompting from the interviewer suggesting the possibility of M and P having the same value did not alter this conviction. For example:

I: What is your answer to question 12?
F: *Never.*
I: Why?
F: *As M cannot equal P.*
I: But what if M and P had the same value, say 5?
F: *If they had the same value, then the same letters would be used, not different ones.*

The second group in this category showed responses similar to those used when solving equations, which happened to result in the correct solution. This procedure involved crossing off the L's and N's from both sides, which leaves M = P. Again, this is indicative of students working within a restricted or closed system related to the students' empirical reality.

Category b) responses indicated that the *values* of M and P must be the same if the statement is to be true, but no actual substitutions were made. This is highlighted in the following extract:

I: How do you do Question 12?
P: *With N being on both sides that means that those two are equal because they would be constants or they would have the same value, being the same letter...the other two* [meaning the L's] *would have the same value, so it would depend on the M and P, their values so it would be sometimes...it would depend on the values of M and P.*
I: Can you make some statement about the values of M and P? Can you write down anything further there?
P: *For the righthand side to equal the lefthand side, M would have to equal P for all values.*

CONCLUSION

Higher level interpretation of a letter and the cognitive processes associated with these has been the focus of this study. The concept of a variable which is generally considered to be an advanced interpretation requires the development of quite complex reasoning skills, different from those needed, say, when a pronumeral is interpreted as a generalised number. Empirical data, as well as quantitative data, from this study has supported this view. An analysis carried out on the original 278 written responses to the three Küchemann test items which differentiated these levels of thinking revealed that only 9% of students could correctly answer all three questions, whereas 35% of students were wrong on all three questions.

This study has also highlighted a number of issues which relate to the mental development associated with high level cognitive skills. These can be summarised as follows:

1) A response which appears limited to working within a given system is indicative of a lower order processing skill. Manipulation of symbols without regard to potential restrictions is one such example.

2) The ability to account for possibilities and the consequent limitations, however, suggests the presence of higher order functioning.

3) There appear to be important subcategories to higher order functioning.

Finally, this study has provided insight into how students' responses to certain algebra questions might be evaluated. In particular, it offers a useful framework from which to consider why certain items are seen as more difficult. Hopefully, once more is known, effective teaching strategies can then be established so that higher order responses might be within the grasp of more students.

Acknowledgement

Our thanks to Dr Dietmar Küchemann for the use of his items.

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